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LETTER TO THE EDITOR

Moment of inertia of doubly connecting bonds in two-dimensional bond percolation

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Abstract. By applying a theorem from conformal field theory to bond percolation in two dimensions we obtain an equation for the second moment I of bonds which doubly connect a percolation cluster: $\langle I \rangle = A(5\sqrt{3}/27\pi^2)(p-p_c)^{-2}$. The equation is exact as the occupancy probability p approaches the critical probability p_c from either side. A is a calculable lattice dependent number which for a square lattice equals $\frac{1}{2}$.

Percolation models are used to understand properties of random systems; in particular, random systems which, at a certain concentration of some quantity, make a transition from a disconnected to a connected phase. The connectivity of percolation clusters has therefore been a subject of considerable interest (Pike and Stanley 1981, Coniglio 1982, Stanley and Coniglio 1983). In this letter we will consider doubly connecting bonds. These are pairs of bonds belonging to the same cluster which, if removed together, cause the cluster to which they belong to fall into two disconnected pieces, but if removed separately, leave the cluster connected. We show that the c-theorem sum rule of conformal field theory (Cardy 1988) gives an asymptotically exact equation for the percolation average of the sum of the square of the distances from a fixed bond (at the origin), to all of the bonds which, together with the fixed bond, doubly connect the cluster (figure 1). A given percolation configuration will contribute to this sum only if the lattice edge at the origin is occupied by a bond (a bond must be present in order for it to belong to a pair of doubly connecting bonds). For $p \leq p_c$ every cluster is finite and this definition of doubly connecting bonds is unambiguous. However for $p > p_c$ the edge at the origin may (if it is occupied) belong to the (unique) infinite cluster so that it is necessary to specify boundary conditions. We shall identify the

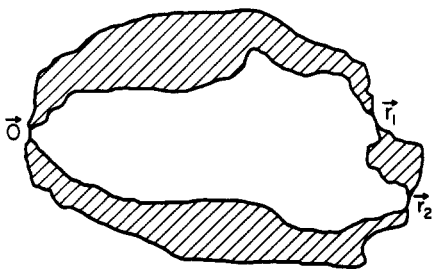

Figure 1. This cluster contributes $I = r_1^2 + r_2^2$ to the moment of inertia of doubly connecting bonds around the origin.

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boundary of the finite lattice with a single point. In the thermodynamic limit, this point becomes the point at infinity. With this boundary condition, 'doubly connecting' has the same meaning for the infinite cluster as it does for finite clusters: a pair of bonds doubly connect the infinite cluster if, when both bonds are removed, the infinite cluster falls into two disconnected pieces (one finite and one infinite), but remains connected when only one is removed. It may be helpful to regard the edge at the origin of the lattice as a battery, whenever it is occupied by a bond. Then the bonds which, along with the bond at the origin, doubly connect the cluster to which they belong (finite or infinite), are just those bonds which carry all the current as it flows from one terminal of the battery to the other. Note that if the battery belongs to the infinite cluster, current may flow through the point at infinity (if the terminals of the battery are connected by separate paths to infinity), it may only flow around some finite loop, or it may not flow at all.

The c-theorem relates the conformal anomaly c, a number which characterises fixed points of the renormalisation group, to the second moment of the connected energy-energy correlation function evaluated in the scaling region, away from criticality:

$$c = 3\pi t^2(2-x_E)^2 \int r^2\langle E(r)E(0)\rangle_c \, \mathrm{d}^2 r. \tag{1}$$

Here $E(r)$ is the local energy density, x_E is the scaling dimension of $E(r)$, $x_E = 2-\nu^{-1}$, and t, proportional to $T-T_c$, is normalised so that in the continuum the reduced Hamiltonian is given by $H_c + t\int E(r)\,\mathrm{d}^2 r$. This theorem is interesting because it relates a quantity that characterises a conformally invariant theory, the conformal anomaly c, to a correlation function in a theory which is not conformally invariant. It therefore extends the predictions of conformal invariance away from the fixed point into the scaling region. We will apply the sum rule to the $q \to 1$ limit of the q-state Potts model which describes bond percolation (Fortuin and Kasteleyn 1972). By interpreting the energy-energy correlation function of the Potts model in terms of bond percolation, one is led to the equation for the moment of inertia of doubly connecting bonds given in the abstract.

The conformal anomaly c (Kadanoff *et al* 1984), and the scaling dimension x_E (den Nijs 1979, Nienhuis 1982) of the q-state Potts model are both known: $c(q) = 1-6(2-g/2)^2/g$, and $x_E = (2+g)/(2-g/2)$, where $q = 2+2\cos(\pi g/2)$, and $(2 \leq g \leq 4)$. Therefore, to apply the c-theorem sum rule, it is only necessary to understand what the second moment of the Potts model energy-energy correlation function corresponds to in terms of percolation. This can be done using a high temperature expansion, the graphs of which correspond to percolation configurations. First consider $p \leq p_c$. The reduced Hamiltonian of the q-state Potts model is given by

$$H = -k\sum_r E(r). \tag{2}$$

Here $E(r) = (q\delta_{\sigma_{r_1},\sigma_{r_2}} - 1)$ where r_1, r_2 label the ends of the bond at r, and the σ are q component spins. The partition function is given by (Wu 1982)

$$Z = \mathrm{Tr}\,\mathrm{e}^{-H} = \mathrm{e}^{-ke}\sum_{\sigma}\prod_{\text{edges}}(1+v\delta_{\sigma_i,\sigma_j}) \tag{3}$$

where $v = e^k - 1$. Each term in the product corresponds to a graph, and each factor $\delta_{\sigma_i,\sigma_j}$ to a bond. Collections of vertices connected by bonds are called clusters. Each cluster contributes a power of q to the trace over σ. Hence, in terms of the occupancy

probability, $p = 1 - e^{-qk}$, the graphical expansion of Z reads

$$Z = e^{(q-1)ke} \sum_G p^b (1-p)^{e-b} q^n \tag{4}$$

where b denotes the number of bonds, and n the number of clusters in G. For $q = 1$, $Z = 1$.

The graphical expansion of $\langle E(r) \rangle$ is well known: when $(q\delta_{\sigma_{r_1},\sigma_{r_2}} - 1)$ is inserted into the trace, the $\delta_{\sigma_{r_1},\sigma_{r_2}}$ allows one to distinguish two kinds of graphs: those in which r_1 and r_2 belong to the same cluster, and those in which they do not. The latter graphs do not contribute, since for these graphs the $\delta_{\sigma_{r_1},\sigma_{r_2}}$ lowers q^n by one power of q, and so equals q^{-1}, so that the contribution of these graphs is proportional to $(qq^{-1} - 1)$, which is identically zero. Therefore

$$\langle E(r) \rangle = (q-1) Z^{-1} e^{(q-1)ke} \sum_G C(r_1, r_2) p^b (1-p)^{e-b} q^n \tag{5}$$

where $C(i, j)$ equals one if i is connected to j and zero otherwise. Thus when q is set equal to one, $\langle E(r) \rangle$ equals, apart from an overall factor of $(q-1)$, the probability that r_1 is connected to r_2 in the percolation problem.

To represent $\langle E(r)E(0) \rangle = \langle q^2 \delta_{\sigma_{r_1},\sigma_{r_2}} \delta_{\sigma_{0_1},\sigma_{0_2}} - q\delta_{\sigma_{r_1},\sigma_{r_2}} - q\delta_{\sigma_{0_1},\sigma_{0_2}} + 1 \rangle$ graphically three types of graphs have to be distinguished. First of all there are the graphs in which r_1 is connected to r_2, and 0_1 is connected to 0_2. For these graphs, both delta functions equal one, so that they contribute

$$(q-1)^2 Z^{-1} e^{(q-1)ke} \sum_G C(r_1, r_2) C(0_1, 0_2) p^b (1-p)^{e-b} q^n \tag{6}$$

to the correlation function. Second, there are the graphs with r_1 connected to r_2 and 0_1 not connected to 0_2, or with 0_1 connected to 0_2 but r_1 not connected to r_2. For these graphs, one of the delta functions equals one, while the other lowers the power of q from the trace by one. Hence the contribution of these graphs is proportional to $(q^2 q^{-1} - qq^{-1} - q + 1)$ which identically equals zero. Finally, there are the graphs with neither r_1 connected to r_2 nor 0_1 connected to 0_2. There are several ways this can happen. If all four vertices belong to different clusters, or if two belong to the same cluster, while the other two belong to different clusters, both delta functions lower q^n by one power of q. The contribution of these graphs is proportional to $(q^2 q^{-2} - qq^{-1} - qq^{-1} + 1 = 0)$, so that these graphs do not contribute either. Lastly there are the graphs with r_1 connected to 0_1 and r_2 connected to 0_2 or vice versa (figure 2). For these graphs

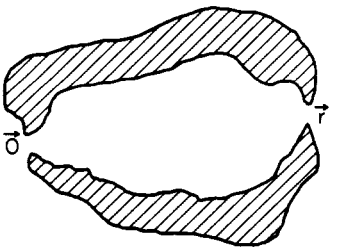

Figure 2. This configuration of two unconnected clusters contributes to the order $(q-1)$ term in the energy-energy correlation function. If bonds are placed at 0 and at r, a doubly connected cluster is formed.

the delta functions equal one another, and have the effect of lowering the power of q^n by one. Hence these graphs contribute

$$(q-1)Z^{-1}\,\mathrm{e}^{(q-1)ke}\sum_{G}U(r,0)\,p^{b}(1-p)^{e-b}q^{n} \tag{7}$$

to the correlation function, where $U(r,0)$ equals one if neither r_1 is connected to r_2 nor 0_1 is connected to 0_2 but r_1 is connected to 0_1 and r_2 is connected 0_2 or vice versa.

The connected energy-energy correlation function, then, consists of two terms. One term, with a coefficient proportional to $(q-1)^2$, corresponds to the connected correlation between r_1 being connected to r_2 on the one hand, and 0_1 being connected to 0_2 on the other. The second term, which has a coefficient proportional to $(q-1)$, corresponds to the probability that, if bonds are put at 0 and r, connecting 0_1 to 0_2, and r_1 to r_2, two previously unconnected clusters become one doubly connected cluster.

The interpretation of the energy-energy correlation function for $p > p_c$ proceeds in the same way as for $p \leqslant p_c$, except that the spins of the Potts model which lie on the lattice boundary must be held fixed to one of their q values in order to break the $Z(q)$ symmetry. Because the boundary spins are fixed, clusters that are connected to the boundary must be distinguished from clusters that are not connected to the boundary, since the former do not contribute any powers of q when the trace over spins is taken. If the lattice boundary is taken to be a single point, every bond connected to the boundary belongs to the spanning cluster. Since the spanning cluster is unique, delta functions in correlation functions play the same role above p_c as they do below: they lower the power of q from the spin trace by one whenever the spins of the delta functions belong to different clusters. Hence the energy-energy correlation has the same interpretation above p_c as it does below.

The c-theorem can now be applied by expanding $c(q)$, $x_E(q)$, and $\langle E(r)E(0)\rangle_c$ around $q=1$ and then equating powers of $(q-1)$. This leads to an infinite number of exact relations for bond percolation in the scaling region. Unfortunately, the relations arising from powers of $(q-1)$ greater than one are of little interest. This is because the expansion of q^n in the energy-energy correlation function introduces factors of n, the total number of clusters on the lattice, into the correlation functions. For example, the term second order in $(q-1)$ gives $\langle nI\rangle - \langle n\rangle\langle I\rangle = B(p-p_c)^{-2}$ where B is a calculable number, and I the moment of inertia of doubly connecting bonds around the origin. If both sides of the above equation are divided by the total number of sites in the lattice, N, to get a meaningful thermodynamic equation, the right side, which is independent of N, vanishes as $N\to\infty$, and one can only say that the number of clusters per site is not correlated to the moment of inertia of doubly connecting bonds.

The order $(q-1)$ term remains. On a regular lattice the c-theorem sum rule reads

$$c(q) = 3\pi t^2(2-x_E(q))^2\left(\frac{a}{e}\right)\sum_{r} r^2\langle E(r)E(0)\rangle_c \tag{8}$$

where a is the total area of the lattice, and $t=(e/a)(k-k_c)$. Using the equations for $c(q)$ and $x_E(q)$ we find $c'(1)=5\sqrt{3}/4\pi$ and $2-x_E=\frac{3}{4}$. To leading order, $k-k_c = \ln(1+2(p_c-p)) = 2(p_c-p)$. Collecting terms we have

$$\sum_{r} r^2\langle E(r)E(0)\rangle_{0(q-1)} = \sum_{r} r^2 \sum_{G} U(r,0)\,p^{b}(1-p)^{e-b} = \left(\frac{a}{e}\right)\frac{5\sqrt{3}}{27\pi^2}(p_c-p)^{-2} \tag{9}$$

where the $\langle E(r)E(0)\rangle_{0(q-1)}$ stands for the term first order in $(q-1)$. The graphs which contribute are those in which two disjoint clusters become doubly connected if bonds

are placed on vacant edges at 0 and r. We could, however, equally well sum over graphs which have bonds at 0 and r that doubly connect a single cluster, as long as the probability for the graph to occur, $p^b(1-p)^{e-b}$, is multiplied by $p^{-2}(1-p)^2$. Moreover we can exchange the order of the sum over r and the sum over G. Then for a given graph, $\Sigma_r r^2 U(r, 0)$ is the sum of the square of the distances from the bond at 0 (if there is one; otherwise the graph does not contribute) to all of the other bonds which, with the bond at 0, doubly connect the cluster to which they belong. Thus we have

$$\langle I\rangle = \sum_G I(G)\, p^b(1-p)^{e-b} = \left(\frac{a}{e}\right)\left(\frac{p}{1-p}\right)^2 \frac{5\sqrt{3}}{27\pi^2}(p_c - p)^{-2} + \text{less singular.} \tag{10}$$

For a square lattice $a/e = \frac{1}{2}$. Also, $p_c = \frac{1}{2}$, so that, in the scaling region, the factor $p^2(1-p)^{-2}$ equals one to leading order in $(p - p_c)$, and so for our purposes may be ignored. Hence for a square lattice,

$$\langle I\rangle = \frac{5\sqrt{3}}{54\pi^2}(p_c - p)^{-2} + \text{less singular.} \tag{11}$$

Monte Carlo simulations on a square lattice give results consistent with the above expression. However statistical errors in the data are too large for these results to be considered a definite confirmation of the predicted equation.

In summary we have found an asymptotically exact expression for the moment of inertia of doubly connecting bonds around the origin. If the telephone network were a percolating system near p_c, this formula gives the distance (on average) that a repairman would have to travel in order to check all the weak links in the network.

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